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1.Introduction

Let A denote the class of functions of the form:

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k, \tag{1}$$

which are analytic in the unit disk $U=\{z:|z|<1\}$. Let $P_\theta(\alpha)$ denote the class of functions of the form:

$$f(z)=e^{-i\theta}+\sum_{k=1}^{\infty}a_kz^k \qquad (-\cos^{-1}\alpha<\theta<\cos^{-1}\alpha),$$

which are analytic and $\mathrm{Re}f(z)>\alpha(0\leq\alpha<1)$ in the unit disk U. We set $P(\alpha)=P_0(\alpha)$.

For a function $f(z)$ in the class A, Salagean ([6]) defined the differential operator $D^n, n\in\mathbb{N}_0=\{0,1,2,3,\cdots\}$, by

$$D^0f(z)=f(z), \qquad D^1f(z)=Df(z)=zf'(z)$$

and

$$D^{n+1}f(z)=D(D^nf(z)) \qquad (n\in\mathbb{N}=\{1,2,3,\cdots\}).$$

If a function $f(z)\in A$ is defined by the form (1), then

$$D^nf(z)=z+\sum_{k=2}^{\infty}k^na_kz^k.$$

Salagean ([6]) also defined the subclass $S^n(\alpha)$ of the class A by

$$S^n(\alpha)=\left\{f(z)\in A:\frac{D^{n+1}f(z)}{D^nf(z)}\in P(\alpha)\right\}$$

for some $\alpha(0\leq\alpha<1)$ and for some $n\in\mathbb{N}_0$. From equalities

$$\frac{D^1f(z)}{D^0f(z)}=\frac{zf'(z)}{f(z)} \quad \text{and} \quad \frac{D^2f(z)}{D^1f(z)}=1+\frac{zf''(z)}{f'(z)},$$

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it follows that $S^0(\alpha) = S^*(\alpha)$ and $S^1(\alpha) = K(\alpha)$, where $S^*(\alpha)$ and $K(\alpha)$ are classes consisting of all starlike and convex (univalent) functions of order α, respectively.

Now we introduce a new class. Let $0 \le \alpha < 1, 0 \le \beta < 1$ and $-\cos^{-1}\beta < \theta < \cos^{-1}\beta$. Then a function $f(z) \in A$ is said to be in the class $C_\theta^n(\alpha,\beta)$ if and only if there is a function $g(z) \in S^n(\alpha)$ and a real number θ such that $\dfrac{D^n f(z)}{e^{i\theta}D^n g(z)} \in P_\theta(\beta)$. Further, we set

$$\overline{C}^n(\alpha,\beta) = \bigcup\{C_\theta^n(\alpha,\beta) : -\cos^{-1}\beta < \theta < \cos^{-1}\beta\}$$

and

$$\underline{C}^n(\alpha,\beta) = \bigcap\{C_\theta^n(\alpha,\beta) : -\cos^{-1}\beta < \theta < \cos^{-1}\beta\}.$$

Kaplan ([3]) defined the class $C_0^1(0,0)$ of close-to-convex functions, and Libera ([4]) defined the class $C_\theta^1(\alpha,\beta)$ of close-to-convex functions of order β and type α. Goodman and Saff ([2]) defined the class $\underline{C}^1(0,0)$, and showed the result $C^1(0,0) = K(0)$ without its proof. The new class $\overline{C}^n(\alpha,\beta)$ is a generalization of the class of close-to-convex functions of order α and type β. With virtue of Lemma 1, Theorems 1 and 2, a function in the class $\overline{C}^n(\alpha,\beta)$ is said to be a *close-to-$S^n(\alpha)$ function of order β*, or a *close-to-S^n function of order β and type α*. A function $f(z)$ in the class $\overline{C}^0(\alpha,0)$ (or $\overline{C}^1(\alpha,0)$) is, respectively, known as a close-to-star function of type α (or a close-to-convex function of type α).

2. Preliminaries

To get our results, we need some lemmas as follows.

Lemma A (MacGregor [5]). *Let $0 \le \alpha < 1$. Then $K(\alpha) \subset S^*(\phi)$, where*

$$(2) \qquad \begin{cases} \phi \equiv \phi(\alpha) = \dfrac{1-2\alpha}{2(2^{1-2\alpha}-1)} & (\alpha \neq \frac{1}{2}) \\ \phi \equiv \phi(\frac{1}{2}) = \dfrac{1}{2\log 2} & (\alpha = \frac{1}{2}). \end{cases}$$

The value of ϕ satisfies that

$$\max\{\alpha, \frac{1}{2}\} < \phi(\alpha) < 1 \qquad (0 \le \alpha < 1)$$

Lemma B (Salagean [6]). *Let $0 \le \alpha < 1$ and $n \in \mathbb{N}_0$. Then $S^{n+1}(\alpha) \subset S^n(\phi(\alpha))$, where $\phi(\alpha)$ is given by* (2).

For $0 \le \alpha < 1$ and $\phi(\alpha)$ defined by (2), let $\{\phi_p\}_0^\infty$ be a sequence defined by mathematical induction as follows:

$$(3) \qquad \phi_0 = \alpha, \quad \phi_{p+1} = \phi(\phi_p) \qquad (p \in \mathbb{N}_0).$$

The sequence $\{\phi_p\}$ satisfies that

$$\max\{\alpha, \frac{1}{2}\} < \phi_1 < \cdots < \phi_p < \phi_{p+1} < \cdots < 1, \qquad \phi_p \to 1 \ (p \to \infty).$$

We get easily the following lemma with virtue of Lemma B.

Lemma 1. *Let $n \in \mathbb{N}_0, p \in \mathbb{N}, 0 \leq \alpha < 1$ and let $\{\phi_p\}$ be defined by* (3). *Then*

$$S^{n+p}(\alpha) \subset S^n(\phi_p) \subsetneq S^n(\alpha).$$

Lemma C (Bernardi [1]). *Let $0 \leq \alpha < 1$, $\operatorname{Re} c \leq \alpha$ and $f(z) \in P(\alpha)$. Then*

$$\left|\frac{f'(z)}{f(z)-c}\right| \leq \frac{2(1-\alpha)}{(1-|z|)\{1-\operatorname{Re} c+(1-2\alpha+\operatorname{Re} c)|z|\}}.$$

3. Main results

Theorem 1. *Let $n \in \mathbb{N}_0, 0 \leq \alpha < 1$ and $0 \leq \beta < 1$. Then $S^n(\alpha) = \underline{C}^n(\alpha,\beta) \subsetneq C^n_\theta(\alpha,\beta)$ for all real $\theta(|\theta| < \cos^{-1}\beta)$.*

Proof. If $f(z) \in S^n(\alpha)$, then there is a function $g(z) \equiv f(z) \in S^n(\alpha)$ such that $\frac{D^n f(z)}{e^{i\theta}D^n g(z)} \equiv e^{-i\theta} \in P_\theta(\beta)$ for $0 \leq \beta < 1$ and real $\theta(|\theta| < \cos^{-1}\beta)$, which proves $S^n(\alpha) \subset \underline{C}^n(\alpha,\beta)$. Conversely, suppose $f(z) \in \underline{C}^n(\alpha,\beta)$ for $0 \leq \alpha < 1$ and $0 \leq \beta < 1$. Then for all real $\theta(|\theta| < \cos^{-1}\beta)$ there is a function $g(z) \equiv g_\theta(z) \in S^n(\alpha)$ such that $\frac{D^n f(z)}{e^{i\theta}D^n g(z)} \in P_\theta(\beta)$. Applying the function $w(z)$ defined by

$$w(z) = \frac{D^n f(z)}{e^{i\theta}D^n g(z)} + 1 - e^{-i\theta} \in P(1-\cos\theta+\beta) \quad (0 < \beta < 1)$$

to Lemma C, we have

$$\left|\frac{D^{n+1}f(z)}{D^n f(z)} - \frac{D^{n+1}g(z)}{D^n g(z)}\right| = \left|\frac{zw'(z)}{w(z)+e^{-i\theta}-1}\right| \leq \frac{2(\cos\theta-\beta)|z|}{(1-|z|)\{\cos\theta+(\cos\theta-2\beta)|z|\}}$$

and therefore

$$\begin{aligned}\operatorname{Re}\frac{D^{n+1}f(z)}{D^n f(z)} &\geq \operatorname{Re}\frac{D^{n+1}g(z)}{D^n g(z)} - \frac{2(\cos\theta-\beta)|z|}{(1-|z|)\{\cos\theta+(\cos\theta-2\beta)|z|\}} \\ &\geq (1-\alpha)\frac{1-|z|}{1+|z|}+\alpha-\frac{2(\cos\theta-\beta)|z|}{(1-|z|)\{\cos\theta+(\cos\theta-2\beta)|z|\}}.\end{aligned} \tag{4}$$

For fixed $z \in U$, the value of the last formula of inequality (4) is larger than α when we choose θ such that the value of $\cos\theta - \beta > 0$ is sufficiently small. This proves $f(z) \in S^n(\alpha)$ and hence $S^n(\alpha) = \underline{C}^n(\alpha,\beta)$ for $0 < \beta < 1$. For $\beta = 0$, we define the function $p(z) \in P(0)$ by

$$p(z)\cos\theta - i\sin\theta = \frac{D^n f(z)}{e^{i\theta}D^n g(z)} \in P_\theta(0)$$

Then we have

$$\begin{aligned}\operatorname{Re}\frac{D^{n+1}f(z)}{D^n f(z)} &\geq \operatorname{Re}\frac{D^{n+1}g(z)}{D^n g(z)} - \left|\frac{D^{n+1}f(z)}{D^n f(z)} - \frac{D^{n+1}g(z)}{D^n g(z)}\right| \\ &\geq \alpha+(1-\alpha)\frac{1-|z|}{1+|z|} - \left|\frac{zp'(z)\cos\theta}{p(z)\cos\theta - i\sin\theta}\right|.\end{aligned} \tag{5}$$

For fixed $z \in U$, the value of the last formula of inequality (5) is larger than α for sufficiently small $\cos\theta > 0$. This proves $f(z) \in S^n(\alpha)$ and hence $S^n(\alpha) = \underline{C}^n(\alpha, 0)$. Finally, we have to prove $S^n(\alpha) \neq C^n_\theta(\alpha, \beta)$, and hence the existence of a function in the class $C^n_\theta(\alpha,\beta) - S^n(\alpha)$ for all real $\theta(|\theta| < \cos^{-1}\beta)$. The function $f_\theta(z) \in A$ defined by $D^n f_\theta(z) = \dfrac{z\{1 + e^{i\theta}(e^{i\theta} - 2\beta)z\}}{(1-z)^{3-2\alpha}}$ is in the class $C^n_\theta(\alpha,\beta)$. Because the function $g(z) \in A$ defined by $D^n g(z) = \dfrac{z}{(1-z)^{2(1-\alpha)}}$ satisfies

$$g(z) \in S^n(\alpha), \qquad \frac{D^n f_\theta(z)}{e^{i\theta} D^n g(z)} = \frac{e^{-i\theta} + (e^{i\theta} - 2\beta)z}{1-z} \in P_\theta(\beta).$$

That $f_\theta(z) \notin S^n(\alpha)$ for any $0 \le \alpha < 1$ and $0 \le \beta < \cos\theta$ is shown as follows. Suppose that $f_\theta(z) \in S^n(\alpha)$ for some $\alpha(0 \le \alpha < 1)$ and some $\beta(0 \le \beta < \cos\theta)$. Since

$$\frac{D^{n+1} f_\theta(z)}{D^n f_\theta(z)} = 2\alpha - 1 - \frac{1}{1 + e^{i\theta}(e^{i\theta} - 2\beta)z} + \frac{3-2\alpha}{1-z},$$

hence the inequality

$$\text{(6)} \qquad \begin{aligned} \operatorname{Re}\frac{D^{n+1} f_\theta(-re^{-i\theta})}{D^n f_\theta(-re^{-i\theta})} =& 2\alpha - 1 - \frac{1 + 2\beta r - r\cos\theta}{(1 + 2\beta r - r\cos\theta)^2 + r^2\sin^2\theta} \\ &+ \frac{(3-2\alpha)(1 + r\cos\theta)}{(1 + r\cos\theta)^2 + r^2\sin^2\theta} > \alpha \end{aligned}$$

has to hold true for some $\alpha(0 \le \alpha < 1)$, some $\beta(0 < \beta < 1)$, all $r(0 \le r < 1)$ and all $\theta(|\theta| < \cos^{-1}\beta)$, and the inequality

$$\text{(7)} \qquad \operatorname{Re}\frac{D^{n+1} f_\theta(-re^{-2i\theta})}{D^n f_\theta(-re^{-2i\theta})} = 2\alpha - 1 - \frac{1}{1-r} + \frac{(3-2\alpha)(1 + r\cos 2\theta)}{1 + 2r\cos 2\theta + r^2} > \alpha$$

has to hold true for some $\alpha(0 \le \alpha < 1), \beta = 0$, all $r(0 \le r < 1)$ and all $\theta(|\theta| < \frac{\pi}{2})$. When $0 \le \alpha < 1$ and $0 < \beta < 1$, we have

$$\lim_{r \to 1-0} \operatorname{Re}\frac{D^{n+1} f_\theta(-re^{-i\theta})}{D^n f_\theta(-re^{-i\theta})} = \alpha - \frac{2\beta(\cos\theta - \beta)}{(1 + 2\beta - \cos\theta)^2 + \sin^2\theta} < \alpha$$

for fixed θ and α, which contradicts the inequality (6). When $0 \le \alpha < 1$ and $\beta = 0$, we have

$$\lim_{r \to 1-0} \operatorname{Re}\frac{D^{n+1} f_\theta(-re^{-2i\theta})}{D^n f_\theta(-re^{-2i\theta})} = -\infty < \alpha$$

for fixed θ and α, which contradicts the inequality (7). This proves $f_\theta(z) \notin S^n(\alpha)$. $\square$

Theorem 2. *Let $n \in \mathbb{N}, 0 \le l \le n-1$ and $0 \le \alpha < 1$. Then*

$$S^{n-1}(\alpha) \subset C_0^n(\alpha,\beta) \qquad (0 \le \beta \le \alpha) \tag{8}$$

and

$$S^l(\alpha) \not\subset \overline{C}^n(\alpha,\beta) \qquad (\alpha < \beta < 1). \tag{9}$$

Proof. Let $f(z) \in S^{n-1}(\alpha)$, and $g(z) = \int_0^z \frac{f(z)}{z}dz$. Then we have

$$zg'(z) = f(z), \quad D^n g(z) = D^{n-1}f(z) \in S^*(\alpha)$$

Therefore there is the function $g(z) \in S^n(\alpha)$ such that $\frac{D^n f(z)}{D^n g(z)} = \frac{D^n f(z)}{D^{n-1} f(z)} \in P(\alpha)$. This proves $S^{n-1}(\alpha) \subset C_0^n(\alpha,\alpha)$ and (8). We define the function $f_\alpha(z) \in A$ by

$$D^{n-1}f_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \in S^*(\alpha) \qquad (0 \le \alpha < \beta < 1).$$

Since $f_\alpha(z) \in S^{n-1}(\alpha)$, we have only to prove $f_\alpha(z) \notin C_\theta^n(\alpha,\beta)$ for all α,β and $\theta (0 \le \alpha < \beta < \cos\theta \le 1)$ to prove (9). If $f_\alpha(z) \in C_\theta^n(\alpha,\beta)$ for some α,β and $\theta (0 \le \alpha < \beta < \cos\theta \le 1)$, then there is a function $g(z) \in S^n(\alpha)$ such that $\frac{D^n f_\alpha(z)}{e^{i\theta} D^n g(z)} \in P_\theta(\beta)$. We define the function $w(z)$ by

$$w(z) = \frac{\{D^{n-1}f_\alpha(z)\}'}{e^{i\theta}\{D^{n-1}g(z)\}'} = \frac{D^n f_\alpha(z)}{e^{i\theta}D^n g(z)} \in P_\theta(\beta).$$

Since

$$D^{n-1}g(z) \in K(\alpha), \qquad \frac{zw'(z)}{w(z)} = \frac{z\{D^{n-1}f_\alpha(z)\}''}{\{D^{n-1}f_\alpha(z)\}'} - \frac{z\{D^{n-1}g(z)\}''}{\{D^{n-1}g(z)\}'},$$

hence we have

$$\begin{aligned}
\mathrm{Re}\frac{zw'(z)}{w(z)} &= \mathrm{Re}\left(1 + \frac{z\{D^{n-1}f_\alpha(z)\}''}{\{D^{n-1}f_\alpha(z)\}'}\right) - \mathrm{Re}\left(1 + \frac{z\{D^{n-1}g(z)\}''}{\{D^{n-1}g(z)\}'}\right) \\
&\le \mathrm{Re}\left(1 + \frac{(1-2\alpha)z}{1+(1-2\alpha)z} + \frac{(3-2\alpha)z}{1-z}\right) - (1-\alpha)\frac{1-|z|}{1+|z|} - \alpha \\
&= 2(1-\alpha)\mathrm{Re}\left(\frac{2z+(1-2\alpha)z^2}{(1-z)\{1+(1-2\alpha)z\}} + \frac{|z|}{1+|z|}\right), \qquad (|z|<1)
\end{aligned}$$

and

$$\mathrm{Re}\frac{-rw'(-r)}{w(-r)} \le -\frac{2(1-\alpha)r}{(1+r)\{1-(1-2\alpha)r\}} \qquad (0 \le r < 1). \tag{10}$$

Otherwise, from the relation $\frac{w(z)+i\sin\theta}{\cos\theta} \in P(\frac{\beta}{\cos\theta})$ and Lemma C, we also have

$$\left|\mathrm{Re}\frac{zw'(z)}{w(z)}\right| \le \frac{2(\cos\theta-\beta)|z|}{(1-|z|)\{\cos\theta+(\cos\theta-2\beta)|z|\}} \qquad (|z|<1),$$

and

$$\text{(11)} \qquad \mathrm{Re}\frac{-rw'(-r)}{w(-r)} \geq -\frac{2(\cos\theta-\beta)r}{(1-r)\{\cos\theta+(\cos\theta-2\beta)r\}} \qquad (0 \leq r < 1).$$

Therefore, with virtue of inequalities (10) and (11), we have

$$\text{(12)} \qquad \frac{1-\alpha}{(1+r)\{1-(1-2\alpha)r\}} < \frac{\cos\theta-\beta}{(1-r)\{\cos\theta+(\cos\theta-2\beta)r\}}$$

for some α, β and $\theta(0 \leq \alpha < \beta < \cos\theta \leq 1)$, and all $r(0 < r < 1)$. Letting $r \to 0$ in the both sides of the inequality (12), we get $\beta \leq \alpha\cos\theta \leq \alpha$, which contradicts $\alpha < \beta$. This proves (9) for $l = n-1$. By Lemma 1, we prove the assertion (9) for $0 \leq l \leq n-1$. □

Many mathematicians have given the class of close-to-convex functions *geometrical* meanings. One of the meanings is that the boundary curve of the image $f(U)$ of the unit disk U by a close-to-convex function $f(z)$ has no "hair pin " bend that exceeds π. Another is that the complex plane minus the image $f(U)$ is the union of closed half- lines such that the corresponding open half-lines are disjoint.

We give the class $\overline{C}^n(\alpha,\beta)$ of close-to-$S^n(\alpha)$ functions of order β *set-theorecal* meanings as follows:

$$\text{(13)} \qquad \begin{cases} S^m(\alpha) \subsetneq S^n(\alpha) = \underline{C}^n(\alpha,\beta) \subsetneq \overline{C}^n(\alpha,\beta) & (0 \leq \alpha < 1, 0 \leq \beta < 1, n < m), \\ S^l(\alpha) \not\subseteq \overline{C}^n(\alpha,\beta) & (0 \leq \alpha < \beta < 1, 0 \leq l \leq n-1), \\ S^{n-1}(\alpha) \subset C_0^n(\alpha,\beta) \subsetneq \overline{C}^n(\alpha,\beta) & (0 \leq \beta \leq \alpha < 1). \end{cases}$$

Putting $n = 1$ and $\beta = 0$ in the last inclusion relation of (13), we have the following Corollary which is well-known.

Corollary. *A starlike function of order α is a close-to-convex of order α.*

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